Characterizing self-organization and coevolution by ergodic invariants

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Abstract

In addition to the emergent complexity of patterns that appears when many agents come in interaction, it is also useful to characterize the dynamical processes that lead to their self-organization. A set of ergodic invariants is identified for this purpose, which is computed in several examples, namely a Bernoulli network with either global or nearest-neighbor coupling, a generalized Bak-Sneppen model and a continuous minority model.

1 Introduction

When a set of agents come in interaction there are, in general, thresholds on the interaction strength above which the whole system self-organizes into distinct patterns of collective behavior. If the dynamics of the agents and their interactions are fixed for all time, the formation of collective patterns is of no concern for the agents themselves. Rather, the identification of patterns belongs to the information compression process of the external observers. However, if the agents themselves are capable of adaptation to the environment, then, the nature of the collective phenomena strongly determines the process of coevolution.

Patterns of collective organization imply a correlation between the configurations in phase space for the components of the system. Therefore quantities like the entropy excess or the statistical complexity[1] [2] [3], may be

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used to characterize the complexity of the patterns that arise as a result of the interactions. However, because the formation of the patterns is related to the nature of the dynamical laws, it seems appropriate to look also for quantities that relate directly to the dynamical process of self-organization. Physically, the most relevant indices must be those that are robust in a probability sense, that is, that are invariant almost everywhere in the support of a physical measure. Therefore in this paper one looks for ergodic invariants and, in particular, for those that emphasize the dynamical relations between the system as a whole and its parts as well as those that provide a dynamical characterization of the collective structures.

Conditional exponents, corresponding to several splittings of the system, when compared with the full Lyapunov exponents, are a measure of the distinction between intrinsic dynamics and the dynamics that arises from the interaction itself. On the other hand the conditional entropies corresponding to cylindrical splittings of the phase space characterize the relative independence of the parts in a collective system. The information provided by the conditional entropies is, in general, not equivalent to the information provided by the conditional exponents.

Another kind of information that is relevant to the characterization of composite dynamical systems is the nature and origin of collective structures. Structures, either temporal or spatial structures, are features that occur at a (time- or space-) scale which is small as compared to the scale of the individual components when in isolation. Temporal structures are found to be related to the variation of the Lyapunov exponents with the interaction strength. Of particular relevance is the critical behavior that occurs in regions where some of the Lyapunov exponents approach zero. This is used to define a structure index for temporal structures.

Conditional exponents, conditional exponent entropies and structure indices are computed in several examples and related to their dynamical features. They seem to relate well to the dynamical features in Bernoulli networks, both with global and nearest-neighbor couplings, and in the generalized Bak-Sneppen model studied in Sect.5.3. As for the last model that is studied, a continuous version of the minority model, it belongs to a class of models where the agents do not interact directly among themselves but only through a common environment. In turn, the environment is a collective variable that the agents themselves create. For this class of models, conditional exponents, in particular, are not relevant and some other invariants,

beyond those developed in this paper, may be needed.

2 Conditional exponents and conditional exponent entropies

A dynamical system lives on the support of some measure μ which is left invariant by the dynamics. An *ergodic invariant* is a dynamical characterization of this measure

$$I\left(\mu\right) = \lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^{T} F\left(f^n x_0\right) \qquad\qquad x_0 \ \mu - a.e.$$

Self-organization in a system concerns the dynamical relation of the whole to its parts. The conditional Lyapunov exponents, introduced by Pecora and Carrol[4] [5] in their study of synchronization of chaotic systems, are quantities that in some sense try to separate the intrinsic dynamics of each component from the influence of the other parts in the system.

Let a mapping $f : M \to M$, with $M \subset R^m$ and an invariant measure μ define a $m-$dimensional dynamical system. The *conditional exponents* associated to the splitting $\Sigma = R^k \times R^{m-k}$ are the eigenvalues $\xi_i^{(k)}$ and $\xi_i^{(m-k)}$ of the limits

$$\lim_{n \to \infty} \left(D_k f^{n*}(x) D_k f^n(x)\right)^{\frac{1}{2n}}$$
$$\lim_{n \to \infty} \left(D_{m-k} f^{n*}(x) D_k f^n(x)\right)^{\frac{1}{2n}}$$

where $D_k f^n$ and $D_{m-k} f^n$ are the $k \times k$ and $m - k \times m - k$ diagonal blocks of the full Jacobian. The conditional exponents are good ergodic invariants.

Lemma. Existence of the conditional exponents as well defined ergodic invariants is guaranteed under the same conditions that establish the existence of the Lyapunov exponents

Proof:

Let μ be an ergodic $f-$invariant measure. Then, Oseledec's multiplicative ergodic theorem, generalized for non-invertible f [6] [7] implies that if the map $T : M \to M_m$ from M to the space of $m \times m$ matrices is measurable

and

$$\int \mu(dx) \log^+ \|T(x)\| < \infty \tag{1}$$

(with $\log^+ g = \max(0, \log g)$) and if

$$T_x^n = T(f^{n-1}x) \cdots T(fx)T(x) \tag{2}$$

then

$$\lim_{n \to \infty} (T_x^{n*} T_x^n)^{\frac{1}{2n}} = \Lambda_x \tag{3}$$

exists μ almost everywhere.

If T_x is the full Jacobian $Df(x)$ and if $Df(x)$ satisfies the integrability condition, the Lyapunov exponents exist $\mu-$a e.

If the Jacobian satisfies (1), then the $m \times m$ matrix formed by the diagonal $k \times k$ and $m - k \times m - k$ diagonal blocks also satisfies the same condition. Therefore conditional exponents too are defined μ-a. e..

Furthermore, under the same conditions, the set of regular points is Borel of full measure and

$$\lim_{n \to \infty} \frac{1}{n} \log \|D_k f^n(x)u\| = \xi_i^{(k)}$$

with $0 \neq u \in E_x^i / E_x^{i+1}$, E_x^i being the subspace of R^k spanned by eigenstates corresponding to eigenvalues $\leq \exp(\xi_i^{(k)})$.

Conditional exponent entropies

For an invariant measure μ absolutely continuous with respect to the Lebesgue measure of M or for measures that are smooth along unstable directions (BRS measures), Pesin's identity[8] states that the sum over positive Lyapunov exponents coincides with the Kolmogorov-Sinai entropy. By analogy one defines the *conditional exponent entropies* associated to the splitting $R^k \times R^{m-k}$ as

$$h_k(\mu) = \sum_{\xi_i^{(k)} > 0} \xi_i^{(k)}$$

$$h_{m-k}(\mu) = \sum_{\xi_i^{(m-k)} > 0} \xi_i^{(m-k)}$$

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These quantities, defined in terms of the conditional exponents, that are good ergodic invariants, are also well-defined ergodic invariants. Here these quantities are defined directly in terms of the conditional exponents. In the next section we will describe an entropy construction in terms of the dynamical refinements of partitions that correspond to the same splitting of the phase space.

A measure of dynamical selforganization

In information theory the mutual information $I(A : B)$ is

$$I(A : B) = S(A) + S(B) - S(A + B)$$

By analogy one defines a *measure of dynamical selforganization* $I(S, \Sigma, \mu)$

$$I(S, \Sigma, \mu) = \sum_{k=1}^{N} \{h_k(\mu) + h_{m-k}(\mu) - h(\mu)\}$$

The sum is over all relevant partitions $R^k \times R^{m-k}$ and $h(\mu)$ is the sum of the positive Lyapunov exponents

$$h(\mu) = \sum_{\lambda_i > 0} \lambda_i$$

$I(S, \Sigma, \mu)$ is also a well-defined ergodic invariant for the measure μ.

The Lyapunov exponents of a dynamical system measure the rate of information production or, from an alternative point of view, they define the dynamical freedom of the system, in the sense that they control the amount of change that is needed today to have an effect on the future. In this sense the larger a Lyapunov exponent is, the freer the system is in that particular direction, because a very small change in the present state will induce a large change in the future. From the point of view of the unit k and of the remaining subsystem, the quantity $h_k(\mu) + h_{m-k}(\mu)$ is therefore the apparent dynamical freedom that they possess (or the apparent rate of information production). The actual rate is in fact $h(\mu)$. Hence $I(S, \Sigma, \mu)$ is a measure of the apparent excess dynamical freedom.

3   Cylindrical partitions and conditional entropies

Consider cylindrical partitions adapted to the splitting $R^k \times R^{m-k}$, namely

$$
\begin{aligned}
\eta^{(k)} &= \left\{ C_1^{(k)}, C_2^{(k)}, \cdots \right\} \\
\eta^{(m-k)} &= \left\{ C_1^{(m-k)}, C_2^{(m-k)}, \cdots \right\}
\end{aligned}
$$

where $C_i^{(k)}$ and $C_i^{(m-k)}$ are k and $m-k$ -dimensional cylinder sets in R^m.

Let now the ζ be a generator partition for the dynamics (f, μ) and define the *conditional entropies* associated to the splitting $R^k \times R^{m-k}$ by

$$
\begin{aligned}
\mathfrak{h}^{(\ell)} &= \sup_{\eta^{(k)}} \lim_{n \to \infty} \frac{1}{n+1} H \left(\zeta \vee f^{-1}\zeta \vee \cdots \vee f^{-n}\zeta \mid \eta^{(k)} \right) \\
\mathfrak{h}^{(\mathfrak{m}-\ell)} &= \sup_{\eta^{(m-k)}} \lim_{n \to \infty} \frac{1}{n+1} H \left(\zeta \vee f^{-1}\zeta \vee \cdots \vee f^{-n}\zeta \mid \eta^{(m-k)} \right)
\end{aligned}
$$

$H \left(\chi \mid \eta \right)$ being

$$
H \left(\chi \mid \eta \right) = - \int_{M/\eta} \sum_i \mu \left(C_i^{(\chi)} \mid \eta \right) \ln \mu \left(C_i^{(\chi)} \mid \eta \right) d\mu
$$

That is, the conditional entropies are the supremum over all cylinder partitions of the sum of the conditional Kolmogorov-Sinai entropies.

The conditional entropies have, in general, a meaning different from the conditional exponent entropies defined before. They might nevertheless be useful for the characterization of relative independence between the components of a complex system. In the following, only the conditional exponent entropies will be used.

4   The structure index

The Lyapunov exponents, as opposed to the conditional exponents, are a global characterization of the dynamics. However they may also be used to extract information, on the relation between the whole system and its parts. If the dynamics of a single isolated unit is known, comparing this with the

spectrum of Lyapunov exponents of the coupled system, information may be obtained on how the collective motion and coherent structures are organized.

A coherent structure (in a collective system) is a phenomenon that operates at a scale very different from the scale of the component units in the system. A structure in space is a feature at a length scale larger than the characteristic size of the components and a structure in time is a phenomenon with a time scale larger than the cycle time of the individual components. A (temporal) *structure index* may then be defined by

$$S = \frac{1}{N} \sum_{i=1}^{N_s} \frac{T_i - T}{T} \tag{4}$$

where N is the total number of components (degrees of freedom) of the coupled system, N_s is the number of structures, T_i is the characteristic time of structure i and T is the cycle time of the isolated components (or, alternatively the characteristic time of the fastest structure). A similar definition applies for a *spatial structure index*, by replacing characteristic times by characteristic lengths.

Structures are collective motions of the system. Therefore their characteristic times are the characteristic times of the separation dynamics, that is, the inverse of the positive Lyapunov exponents. Hence, for the temporal structure index, one may write

$$S = \frac{1}{N} \sum_{i=1}^{N_+} \left(\frac{\lambda_0}{\lambda_i} - 1 \right) \tag{5}$$

the sum being over the positive Lyapunov exponents λ_i. λ_0 is the largest Lyapunov exponent of an isolated component or some other reference value.

The temporal structure index diverges whenever a Lyapunov exponent approaches zero from above. Therefore the index diverges at the points where (in the separation dynamics) long time correlations develop.

5   Examples

5.1 A globally coupled Bernoulli network

The dynamical law is

$$x_i(t+1) = (1-c)f(x_i(t)) + \sum_{j \neq i} \frac{c}{N-1} f(x_j(t)) \tag{6}$$

with $f(x) = 2x \pmod{1}$.

The Lyapunov exponents are

$$\begin{aligned} \lambda_1 &= \log 2 \\ \lambda_i &= \log\left(2\left(1 - \frac{N}{N-1}c\right)\right) \quad \text{with multiplicity} \quad N-1 \end{aligned} \tag{7}$$

Therefore,

$$\begin{aligned} h(\mu) &= \log 2 + (N-1)\log\left(2 - \frac{2N}{N-1}c\right) \quad \text{for} \quad c \leq \frac{N-1}{2N} \\ &= \log 2 \quad \text{for} \quad c \geq \frac{N-1}{2N} \end{aligned} \tag{8}$$

The conditional exponents associated to the splitting $R^1 \times R^{N-1}$ are

$$\xi^{(1)} = \log(2 - 2c) \tag{9}$$

and

$$\begin{aligned} \xi_1^{(N-1)} &= \log\left(2 - \frac{2}{N-1}c\right) \quad \text{once ;} \tag{10} \\ \xi_i^{(N-1)} &= \log\left(2 - \frac{2N}{N-1}c\right) \quad \text{with multiplicity} \quad N-2 \tag{11} \end{aligned}$$

Therefore,

$$I(S, \Sigma, \mu) = N\left(\log\left(1 - \frac{c}{N-1}\right) + \max\left(\log(2 - 2c), 0\right) - \max\left(\log\left(2 - \frac{2Nc}{N-1}\right), 0\right)\right) \tag{12}$$

which, in the limit of large N, becomes

$$\begin{aligned} I(S, \Sigma, \mu) &= \frac{c^2}{1-c} \quad c \leq \frac{N-1}{2N} \\ &= -c \quad c \geq \frac{1}{2} \end{aligned} \tag{13}$$

The variation of $I(S, \Sigma, \mu)$ with the coupling intensity c is plotted in Fig.1. It grows until the synchronization point and then it becomes negative. The transition is discontinuous in the large N limit.

Structures

Except for $c = c_s = \frac{1}{2}\frac{N-1}{N}$ the globally coupled Bernoulli system is uniformly hyperbolic. For coupling strength $c < c_s$ the Lyapunov dimension is N and one expects a BRS- invariant measure absolutely continuous with respect to the Lebesgue measure in R^N. The distribution of the values taken by each unit x_i is essentially flat and, for large N, the mean field seen by one unit has very small fluctuations.

However, as one approaches $c = c_s$ from below, one sees the dynamics organizing itself into synchronized patches, with each patch maintaining an approximately constant phase relation with the other patches. Synchronization and phase locking effects are however not absolutely stable phenomena. In the Figs.2 and 3 one shows the statistics of $|x_i - x_k|$ and $x_i + x_{i+1} - 2x_{i+2}$. For $c = 0.45$ one sees clearly the phenomenon of clustering and synchronization with positive Lyapunov exponents discussed before[9].

For $c < c_s$ all Lyapunov exponents are positive. However, near c_s only one of the Lyapunov exponents is large whereas all the others are nearly zero. That is, there is a fast *separation dynamics* (sensitive dependence to initial conditions) in one direction and very slow separation dynamics in all directions transversal to the fast one. The fast separation direction corresponds to the eigenvector $(1, 1, 1, 1, ..., 1)$

The slow separation dynamics in the transversal directions corresponds to long wavelength effects in phase space which are the most sensitive to boundary conditions and the available phase-space. Then, the slow temporal structures beget non-uniform probability distributions in the linear combinations of the variables that correspond to the slow eigenvalues. In particular, $x_i - x_{i+1}$ corresponds to the eigenvector $(0, ..., 1, -1, 0, ..., 0)$ and $x_i + x_{i+1} - x_{i+2}$ to $(0, ..., 1, 1, -2, 0, ..., 0)$.

The existence of structures near the transition points where one or more Lyapunov exponents approach zero from above is an universal phenomena, whereas the detailed form of the structures depends on the particular nature of the available phase-space.

From (5) and (7) one obtains for the structure index

$$
\begin{aligned}
S &= \frac{N-1}{N}\left(\frac{\log 2}{\log 2\left(1 - \frac{N}{N-1}c\right)} - 1\right) \quad \text{for} \quad \frac{N}{N-1}c < 0.5 \\
S &= 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{for} \quad \frac{N}{N-1}c > 0.5
\end{aligned}
$$

For $\frac{N}{N-1}c > 0.5$ the structure index vanishes because the synchronized motion

is effectively one-dimensional and the characteristic time of the synchronized motion coincides with the characteristic time of the individual units. The structure index is zero both for the uncoupled case and the fully synchronized one and diverges at the synchronization transition (Fig.2).

5.2 Nearest neighbor coupling

Let now

$$x_i(t+1) = (1-c)f(x_i(t)) + \frac{c}{2}\left(f(x_{i+1}(t)) + f(x_{i-1}(t))\right)$$

with $f(x) = 2x$ (mod. 1)

The Lyapunov exponents are

$$\lambda_k = \log\left\{2(1-c) + 2c\cos\left(\frac{2\pi}{n}k\right)\right\}$$

$k = 0, \cdots, n-1$

For the conditional exponents, one is

$$\xi^{(1)} = \log\left\{2(1-c)\right\}$$

and the others are the logarithm of the eigenvalues of the matrix

$$\begin{pmatrix}
2(1-c) & c & 0 & \cdots & 0 \\
c & 2(1-c) & c & \cdots & 0 \\
0 & c & 2(1-c) & c & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & c & 2(1-c)
\end{pmatrix}$$

Fig.4 displays the measure of dynamical selforganization I and the structure index S for this example with $N = 500$. The dynamical behavior in the nearest-neighbor coupled network is more complex than in the globally coupled one, showing a greater diversity of distinct dynamical features. This is reflected in the behavior of the ergodic invariants. For the structure index, in particular, one notices the existence, above $c = 0.5$, of many points where it diverges. These points correspond to the crossing through zero of each individual Lyapunov exponent. Also, the invariant I displays three distinct regions, rather than two as in the globally coupled network.

5.3 A generalized Bak-Sneppen model

As suggested by the examples above, the most interesting events (creation of a large number of structures, for example) occur for particular values of the interaction. Therefore, it would be desirable to have a way to adjust the interaction strength in coevolution models. This is lacking in the original Bak-Sneppen (B-S) model[13]. Also, to define ergodic invariants it is more convenient to have a deterministic dynamics. Notice however that, even in stochastic models, it is possible to define a parameter similar to a Lyapunov exponent, by analyzing the *spread of damage*[11] [12].

The model studied in this section is defined by:

- N species, each one assigned to a lattice point in a one-dimensional lattice, each lattice point standing for an ecological niche. To each lattice point i one assigns a variable x_i (with values between 0 and 1)

At each time step the site with the lowest x_i and its two nearest neighbors are chosen anew according to the law

$$
\begin{aligned}
x_i(t+1) &= (1-c)\, f\left(x_i(t)\right) + \tfrac{c}{2}\left(f\left(x_{i+1}(t)\right) + f\left(x_{i-1}(t)\right)\right) \\
x_{i\pm1}(t+1) &= (1-c)\, f\left(x_{i\pm1}(t)\right) + \tfrac{c}{2}\left(f\left(x_i(t)\right) + f\left(x_{i\mp1}(t)\right)\right)
\end{aligned}
$$

with $f(x) = 2x$ (mod. 1)

When $c = 0$ the function $f(x)$ is a pseudo-random number generator and the model is equivalent to Bak-Sneppen's coarse grained model for evolution. There are however some essential differences:

- B-S is a stochastic model whereas this one is deterministic. The updating function being differentiable, the tools of ergodic theory are applicable and Lyapunov exponents, conditional exponents and entropies may be computed and used to characterize the dynamics.

- The interaction strength between neighboring species may be changed

Parallel versus sequential dynamics

If instead of sequential dynamics (selected by the smallest x_i) one had chosen parallel dynamics, this model would be similar to the nearest-neighbor coupled Bernoulli chain studied before. Then, a large variety of qualitatively different dynamical behaviors are observed, which result from the critical points at different values of c where the Lyapunov exponents cross zero.

When, instead of parallel dynamics, one imposes on the model a sequential dynamics selected by the smallest x_i, what one is really doing is to

introduce a feature that simulates friction or resistance to change in the dynamical system. This has some dramatic consequences for the computation of the exponents in the limit of large N. Instead of a tangent map matrix of the type described before one has now the product of matrices which have ones on the diagonal almost everywhere and only one non-trivial 3×3 block. Therefore the Lyapunov exponents become, on the average

$$
\begin{array}{rcll}
\lambda & \approx & \log{(2)}^{\frac{3}{N}} & \frac{N}{3} \quad \text{times} \\
\lambda' & \approx & \log{\left(2\left(1 - \frac{3}{2}c\right)\right)}^{\frac{3}{N}} & \frac{2N}{3} \quad \text{times}
\end{array}
$$

Therefore, for large N, all the Lyapunov exponents approach zero independently of any other dynamical characteristics and the system is near criticality. Therefore the fact that this type of system appears poised on the edge of criticality is a consequence of the type of sequential dynamics that is chosen.

For the structure index, in addition to the divergence effect obtained in the limit of large N, one has other features that arise from the interaction between the species, namely

$$
\begin{array}{rcl}
c < \frac{1}{3} & S & = & \dfrac{\frac{N}{3\log 2} - \tau_0}{3\tau_0} + 2\dfrac{\frac{N}{3\log 2\left(1 - \frac{3}{2}c\right)} - \tau_0}{3\tau_0} \\[2ex]
c > \frac{1}{3} & S & = & \dfrac{\frac{N}{3\log 2} - \tau_0}{3\tau_0}
\end{array}
$$

where τ_0 is a reference characteristic time (characteristic time of the individual dynamics or characteristic time of the fastest structure). For definiteness one chooses $\tau_0 = \frac{1}{3}\log 2$.

One sees that, besides the overall critical behavior arising from the sequential dynamics, there is additional critical behavior at $c = \frac{1}{3}$ arising from the interactions in the coupled block of three species.

The average conditional exponents are

$$
\begin{array}{rcl}
\mu^{(1)} & \approx & \log{\left(2\left(1 - c\right)\right)}^{\frac{3}{N}} \\[2ex]
\mu^{(N-1)} & \approx & \dfrac{\log{\left(2\left(1 - \frac{1}{2}c\right)\right)}^{\frac{3}{N}}}{\log{\left(2\left(1 - \frac{3}{2}c\right)\right)}^{\frac{3}{N}}}
\end{array}
$$

and the measure of dynamical self-organization is

$$
\begin{array}{rcrcl}
c < \frac{1}{3} & & I & = & 3\left(\log{\left(1 - c\right)} + \log{\left(1 - \frac{c}{2}\right)} - \log{\left(1 - \frac{3}{2}c\right)}\right) \\
\frac{1}{2} > c > \frac{1}{3} & & I & = & 3\left(\log 2 + \log{\left(1 - c\right)} - \log{\left(1 - \frac{1}{2}c\right)}\right) \\
1 > c > \frac{1}{2} & & I & = & 3\log{\left(1 - \frac{c}{2}\right)}
\end{array}
$$

The values of the structure index and the self-organization measure for the generalized Bak-Sneppen model are plotted in Fig.5.

Not only the ergodic invariants, but also variables like the barrier size, show a marked dependence on the coupling strength. Numerically computed barrier sizes are shown in Fig.6.

One sees that from $c = 0$ to $c = \frac{1}{3}$ the barrier stays close to the original B-S value. Then, after $c = \frac{1}{3}$ (the additional critical point of the structure index) it grows to a plateau above 0.95. Eventually, as c increases further, the barrier size comes back to the original B-S value. This is probably related to the fact that for large values of the coupling, when neighboring species become synchronized, there is an almost random evolution of three-species blocks.

The scaling of avalanche sizes is another useful characterization in models of this type. The avalanche size is defined, as usual, by the number of time steps below a fixed threshold. With thresholds for the avalanche definition at 0.65 in the first case and 0.95 in the second, a comparison was made of the $c = 0$ and the $c = 0.5$ cases. In both cases a power law is obtained, $N(s) \sim s^{-\alpha}$, with a 20% larger exponent α in the $c = 0.5$ case.

5.4   A continuous minority model

The minority model introduced by Challet and Zhang,[14], inspired in Brian Arthur's bar model[15], as well as most market models, belongs to a class of models in which the agents do not interact directly among themselves, but only through a common environment. On the other hand, the environment is a collective functional that the agents themselves create. This kind of inter-agents interaction, through an external medium, that they themselves collectively create, has some specific dynamical consequences.

Here one studies a continuous version of the minority model. For each one of N agents there is a variable x_i with values in the interval $[0, 1)$. The average of the values of x_i defines a "mean field" $\overline{x}(t)$ at time t

$$\overline{x}(t) = \frac{1}{N} \sum_i x_i(t)$$

The time evolution of each x_i is only a function of $\overline{x}$ at times $t, t-1, \cdots, t - M$. Let $c \in [0, 1)$ (called the *cut*) define a partition of the interval into

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$\{A = [0, c), B = [c, 1)\}$. At each time t if $x_i(t)$ is in one of these intervals and $\overline{x}(t)$ is in the other, agent i wins a point, otherwise it wins nothing. Hence one has the dynamics

$$x_i(t + 1) = f_i\left(\overline{x}(t), \cdots, \overline{x}(t - M)\right)$$

and a payoff

$$m_i(t + 1) = m_i(t) + \frac{1}{2}\left(1 - \text{sign}\left\{(\overline{x}(t) - c)\left(x_i(t) - c\right)\right\}\right)$$

Each agent has his own function f_i, called the *strategy* of agent i. The time-delayed law may be converted into a single map in a $(M + 1)N$ space

$$\begin{pmatrix} x_i(t) \\ x_i(t-1) \\ \vdots \\ x_i(t-M) \end{pmatrix} \rightarrow \begin{pmatrix} f_i\left(\overline{x}(t), \cdots, \overline{x}(t-M)\right) \\ x_i(t) \\ \vdots \\ x_i(t-M+1) \end{pmatrix}$$

It is easy to see that, for models of this kind, the conditional exponents play a negligible role. Take for example the case $M = 0$. Consider the Jacobian

$$J_p = \frac{\partial x_i(t + p)}{\partial x_j(t)}$$

The eigenvalues of $J^T J$ are $N - 1$ zeros and one

$$\mu = N\left(\frac{1}{N^2}\sum_i f_i'^2(t+p)\right)\left(\frac{1}{N}\sum_i f_i'(t+p-1)\right)^2 \cdots \left(\frac{1}{N}\sum_i f_i'(t)\right)^2$$

Taking the $p \to \infty$ limit in

$$\lim_{p\to\infty} \log\left(J^T J\right)^{\frac{1}{2p}}$$

one obtains a single non-trivial Lyapunov exponent

$$\lambda = \lim_{p\to\infty}\frac{1}{p}\log\left\{\left(\frac{1}{N}\sum_{i=1}^{N} f_i'(t+p-1)\right)\cdots\left(\frac{1}{N}\sum_{i=1}^{N} f_i'(t)\right)\right\}$$

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For the conditional exponents

$$\begin{aligned}
\xi_i^{(1)} &= \lim_{p\to\infty} \tfrac{1}{p} \log\left\{\left(\tfrac{1}{N}f_i'(t+p)\right)\cdots\left(\tfrac{1}{N}f_i'(t)\right)\right\} \\
\xi_i^{(N-1)} &= \lim_{p\to\infty} \tfrac{1}{p} \log\left\{\left(\tfrac{1}{N}\sum_{j\neq i}f_j'(t+p-1)\right)\cdots\left(\tfrac{1}{N}\sum_{j\neq i}f_j'(t)\right)\right\}
\end{aligned}$$

Then, for a sufficiently large number of agents, the conditional exponent $\xi_i^{(1)}$ cannot be positive and the self-organization measure will always vanish in the large N limit. The situation does not change if a non-zero memory size M is considered. Then the conditional exponent $\xi_i^{(1)}$ is computed by the product of blocks of the form

$$\begin{pmatrix}
\tfrac{1}{N}f_i^{(1)} & \tfrac{1}{N}f_i^{(2)} & \cdots & \tfrac{1}{N}f_i^{(M+1)} \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0
\end{pmatrix}$$

where $f_i^{(k)}$ denotes the k-argument derivative of f_i .

For large N the quantities $\tfrac{1}{N}f_i^{(k)}$ are very small and once again the conditional exponent $\xi_i^{(1)}$ cannot be positive. The vanishing of the $I(S,\Sigma,\mu)$ invariant is easy to understand from the qualitative interpretation given in Sect.2. For models of this type, the contribution of each individual agent to his own evolution is extremely small and therefore the difference between $h_k(\mu) + h_{N-k}(\mu)$ and $h(\mu)$ must be negligible.

In the original (discrete) minority model each agent is equipped with several strategies choosing, at each time, the one with the best virtual record. Here each agent has a single strategy which however may be changed according to the following scheme:

- At the start all strategies are chosen at random from a pool of functions;

- After each 10 time steps, the 10 worst performing strategies (in the last 10 steps) are selected for replacement. The three worst ones are replaced by three new strategies chosen at random from the function pool and the remaining seven are replaced by strategies that copy the seven best ones with a small random error.

For definiteness consider the functions to be linear regressions with coef-

ficients $\alpha_k^{(i)}$ taken at random from the interval $[-K, K]$.

$$x_i(t+1) = \sum_{k=0}^{M} \alpha_k^{(i)} \overline{x}(t-k)$$

For sufficiently large K this system self-organizes into configurations away from random choice. The most relevant parameter to track the system behavior is

$$P(t) = \sum_i m_i(t)$$

the total payoff at time t. In the following table one compares the average and standard deviations of P for random choice of the x_i's in the interval $[0, 1)$ with those obtained from numerical simulations of the model for several values of K and cut c and $M = 2$.

Cut	$\overline{P}_{\text{rand}}$	$\sigma(P)_{\text{rand}}$	K	$\overline{P}$	$\sigma(P)$	$\sum_i \lambda_i$
0.6	0.4	0.049	1	0.494	0.1	
			1.5	0.49	0.099	
			2	0.486	0.096	
0.7	0.3	0.0459	0.5	0.495	0.186	-4.38
			1	0.499	0.17	
			1.5	0.496	0.165	-1.13
			2	0.493	0.157	-1.05
			4	0.483	0.15	
			5	0.481	0.157	-2.2

$\overline{P}$ and $\sigma(P)$ are the average and standard deviations of $P(t)$ and $\overline{P}_{\text{rand}}$ and $\sigma(P)_{\text{rand}}$ the values that are obtained for random choice of the x_i's in the interval $[0, 1)$.

One sees that the average value of P is systematically larger than the random value, but with a much larger standard deviation. Of particular significance is the actual probability distribution of this global variable. In Figs.7 and 8 one shows this distribution for $c = 0.6$ and $c = 0.7$. Instead of a Gaussian distribution with a small standard deviation, that would be obtained in the random case, the model organizes itself to have a asymmetrical distribution, with an increased average value, which for large cuts splits into two peaks

The dynamics of the total payoff $P(t)$

$$P(t) = \sum_i \frac{1}{2} \left(1 - \text{sign} \left\{ (\overline{x}(t) - c) \left(\text{mod} \left(\sum_{k=1}^{M+1} \alpha_k^{(i)} \overline{x}(t-k), 1 \right) - c \right) \right\} \right)$$

depends only on the dynamics of the average values

$$\begin{pmatrix} \overline{x}(t) \\ \overline{x}(t-1) \\ \vdots \\ \overline{x}(t-M) \end{pmatrix} \rightarrow \begin{pmatrix} \sum_{k=0}^{M} \frac{1}{N} \sum_i \alpha_k^{(i)} \overline{x}(t-k) + \frac{1}{N} \sum_i \theta_i(t) \\ \overline{x}(t) \\ \vdots \\ \overline{x}(t-M+1) \end{pmatrix}$$

where the integers $\theta_i(t) \in Z$ originate from the mod.1 operation in the dynamics of the x_i's. It is the dynamics of the average values that determines the global behavior of the system. The Lyapunov exponents of this dynamics is computed from the Jacobian matrices

$$\begin{pmatrix} \frac{1}{N} \sum_i \alpha_0^{(i)} & \frac{1}{N} \sum_i \alpha_1^{(i)} & \cdots & \frac{1}{N} \sum_i \alpha_M^{(i)} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Typically, in the (statistically-) stable state of the model, the Lyapunov exponents are all negative. They were computed for typical configurations obtained after many iterations for several values of K at $c = 0.7$. This is the meaning of the last column in the table above. Although the Lyapunov exponents are negative, the dynamics is non-trivial. Without the non-linear effect of the θ_i's the system would stabilize in a fixed point. Instead, for large N, it has very many periodic orbits and the evolution mechanism, emphasizing those that have a higher payoff for each cut, acts a selection mechanism that drives the system to a particular class of orbits. In the Fig.9 one shows the typical time-behavior of the mean value $\overline{x}(t)$ after many iterations of the model. The fluctuations that are observed reflect not only the high period of the orbits but also the change of dynamics imposed by the selection mechanism that is operating all the time.

Thus, the dynamics of this continuous minority model becomes well understood. However the role of the ergodic invariants discussed in Sects. 2-4 is

not so clear. On the one hand, as explained above, the "mean-field" dynamics of the model makes the conditional exponents trivial. On the other hand the structure index invariant is suited mainly to detect structural transitions when a parameter is varied. Here however one finds out that the dynamical structure is largely independent of the parameters that have been explored (K and c). The structure index might however become useful if this model is embedded in some larger class of models.

Another variable that is relevant in the study of collective model of this type is the survival time of each strategy, that is the time interval until it follows in one of the (locally) worst strategies. From the simulations that were performed an exponential behavior ($\exp(-\xi t)$) is obtained, rather than a power law as in the avalanche sizes of the generalized Bak-Sneppen model. Also, the exponent is nearly constant, $\xi \simeq 0.004$, in the range of parameters that was explored

6   Remarks and conclusions

A fascinating aspect of complex systems, and even more of complex adaptive systems, is that the behavior of the whole is so different from and richer than the behavior of the parts. For lack of a precise theory of collective behavior, all kinds of new features that appear in the whole are called *emergent properties*. From simple systems with simple rules, complex macropatterns emerge. To understand why this is so and what universal features, if any, underlie this phenomenon is a challenging task. It is also of practical importance because emergence is ubiquitous in the universe around us.

An almost general rule, in the emergence of macropatterns, is the formation, through interactions, of subassemblies which combine with similar subassemblies, with the structure at each level constraining what emerges at the next level. The nature of the interactions between the agents, and then between the subassemblies, is the key to the understanding of the macropatterns. This is the reason why, to understand emergence and self-organization, it is essential to characterize the interaction dynamics and, in particular, the robust properties of this dynamics, that is, its ergodic properties.

One dynamical effect, identified in this paper, which generates collective patterns is the approach to zero from above of one of the Lyapunov exponents of the global system. The question is how frequently should one expect this

situation to occur. Two basic mechanisms were identified:

- When there is a natural limitation on the range of values that the state variable of each individual agent can take, the coupling must be convex, as in the examples studied in this paper. Then, the convex coupling leads to an overall contracting effect and Lyapunov transitions are expected when the coupling increases. In spatially extended systems, for example, even if the interaction law does not change, a change in density would imply an effective coupling increase. Therefore in a evolving system where the number of agents changes in time (but the available space remains fixed), effects of the type described here might be expected to arise when the population density changes.

That, at the transition regions between chaos and order, evolving systems display interesting structural properties was suggested in the past by several authors[16] [17]. Why some natural systems might have evolved to such narrow regions in parameter space is, to a large extent, an open question. The density-dependent increase of the effective interaction and the contracting effect implied by the convex coupling, when the amount of available phase-space remains constant, is a dynamical mechanism that might explain, in some cases, the evolution towards the transition regions.

- Another mechanism leading to Lyapunov exponents that are positive but close to zero, occurs when agents with sensitive dependent dynamics interact via a friction mechanism. This resistance to change has as a consequence that only the agents under the largest stress will be allowed to evolve. For a large number of agents, this sequential dynamics leads to an effective Lyapunov exponent close to zero. This is a situation that seems to occur in many examples of what has been called *self-organized criticality*. These systems appear poised on the edge of criticality as a consequence of this type of sequential dynamics.

The approach to zero of the Lyapunov exponents corresponds to the divergent points of the (temporal) *structure index*. Another important characterization of the collective system is obtained by invariants constructed from the *conditional exponents*. These are quantities that distinguish the intrinsic dynamics of each agent from the influence of the other parts in the system. In particular the *measure of dynamical selforganization*, discussed in Sect.2, characterizes the excess dynamical freedom that would be perceived by the agents themselves.

Structure indices and the invariants constructed from the conditional ex-

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ponents and conditional entropies do not, however, exhaust the parameters needed to characterize the dynamical and probabilistic behavior of collective systems. Scaling exponents, for example, are currently used and, as seen in the generalized Bak-Sneppen model, may be related to the other invariants. A technique which might also be useful, in the future, is the generalized spectral decomposition, in particular the relation between the spectrum of the Koopman operator of individual agents with the spectrum of the whole system.

Figure captions

Fig.1 - Structure index and self-organization measure for a globally coupled Bernoulli network

Fig.2 - Distribution of the variable $|x_i - x_j|$ for several values of the coupling and $N = 100$

Fig.3 - Distribution of the variable $x_i + x_{i+1} - 2x_{i+2}$ for several values of the coupling and $N = 100$

Fig.4 - Structure index and self-organization measure for a nearest-neighbor coupled Bernoulli network

Fig.5 - Structure index and self-organization measure for the generalized Bak-Sneppen model

Fig.6 - Barrier size for several values of the coupling in the generalized Bak-Sneppen model

Fig.7 - Distribution of P for $c = 0.6$

Fig.8 - Distribution of P for $c = 0.7$

Fig.9 - Typical time behavior of the mean value in the continuous minority model

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Fig.1
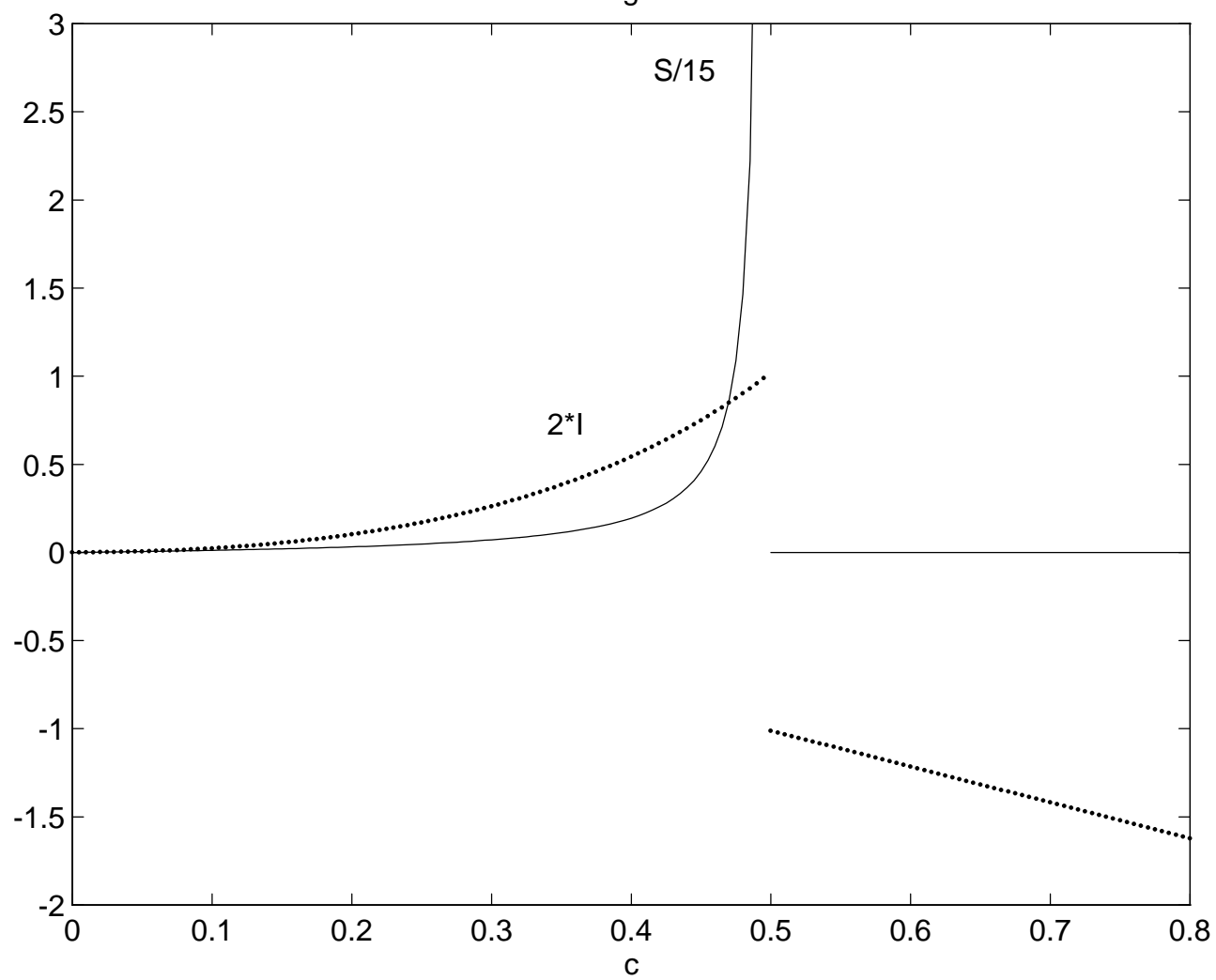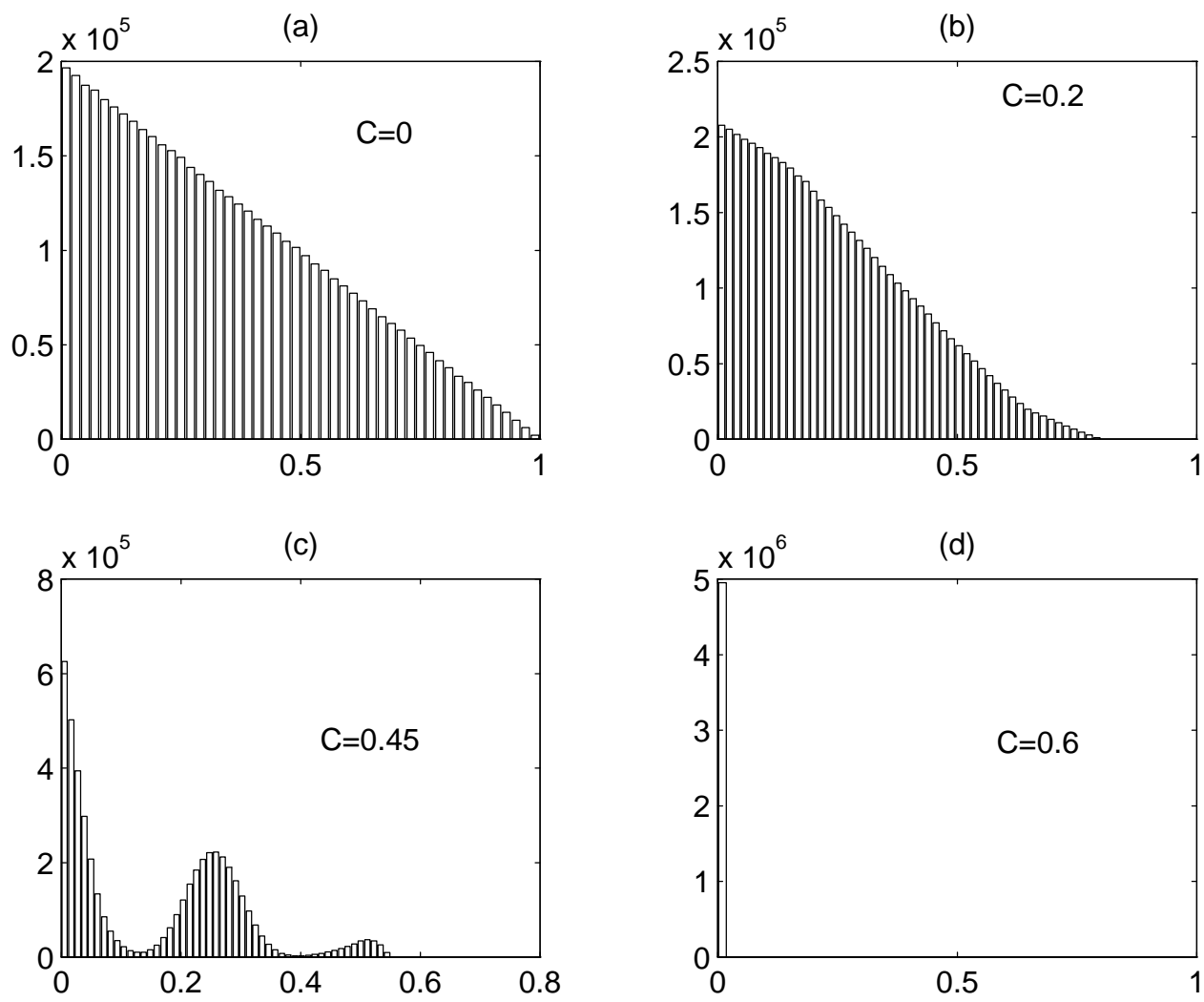
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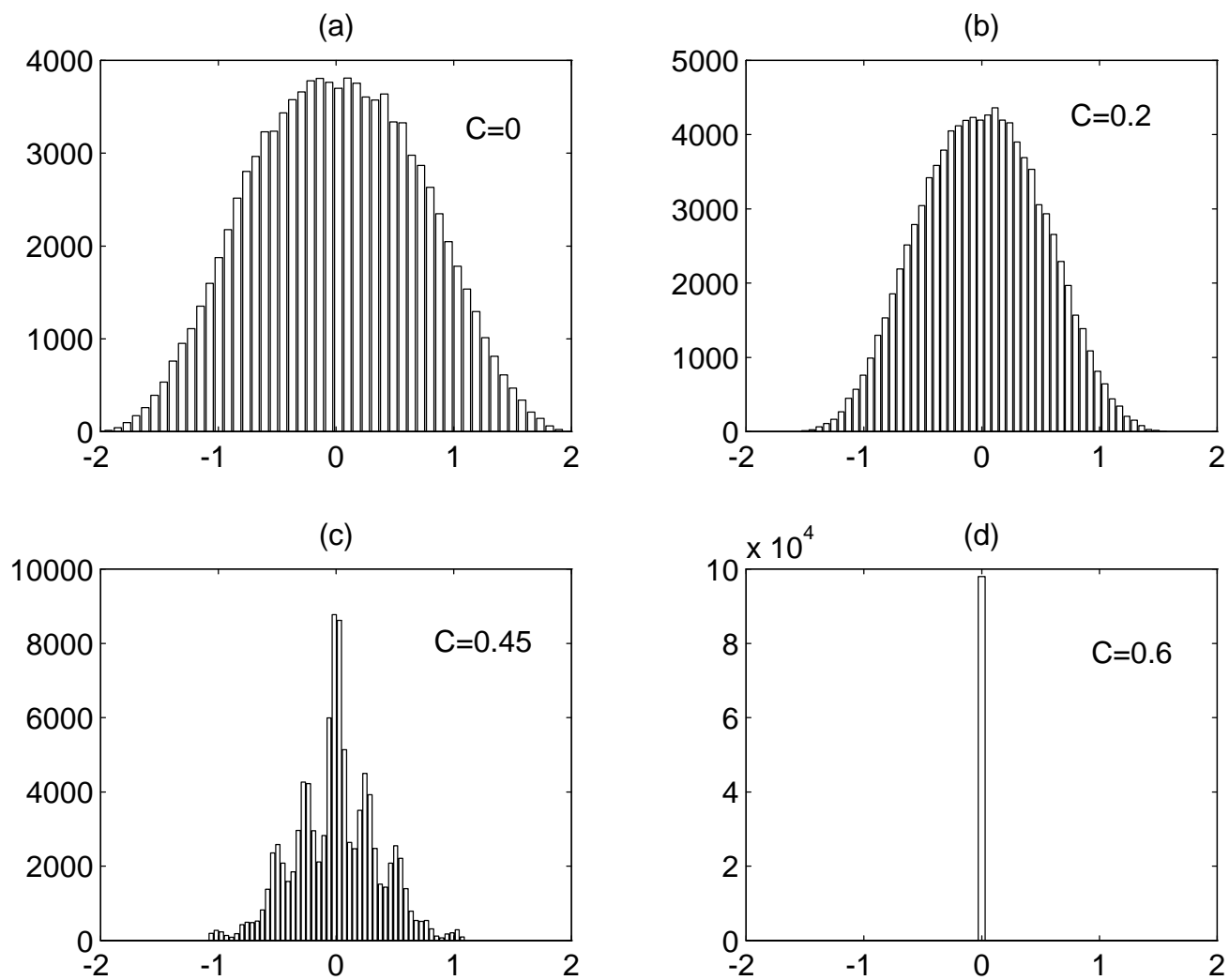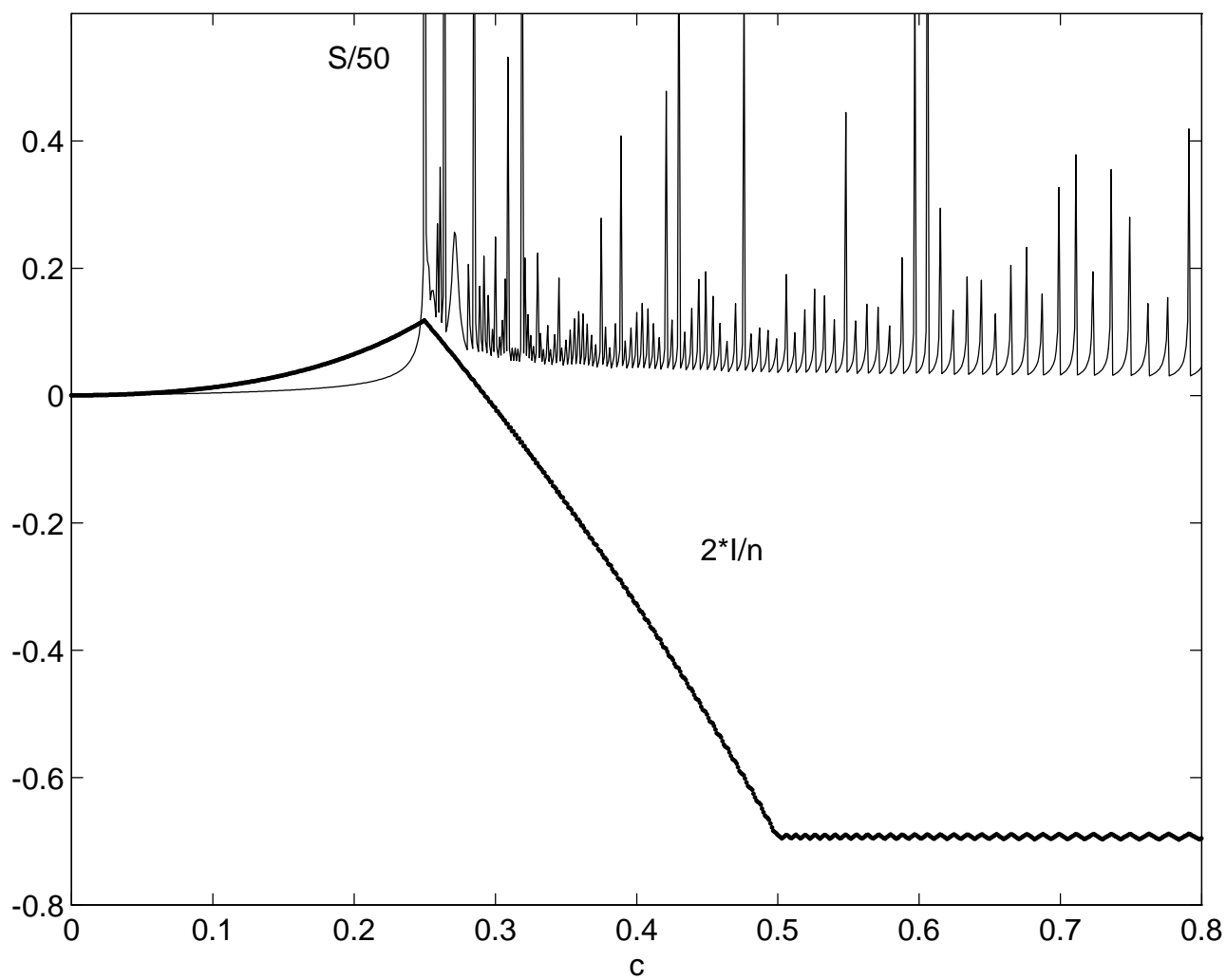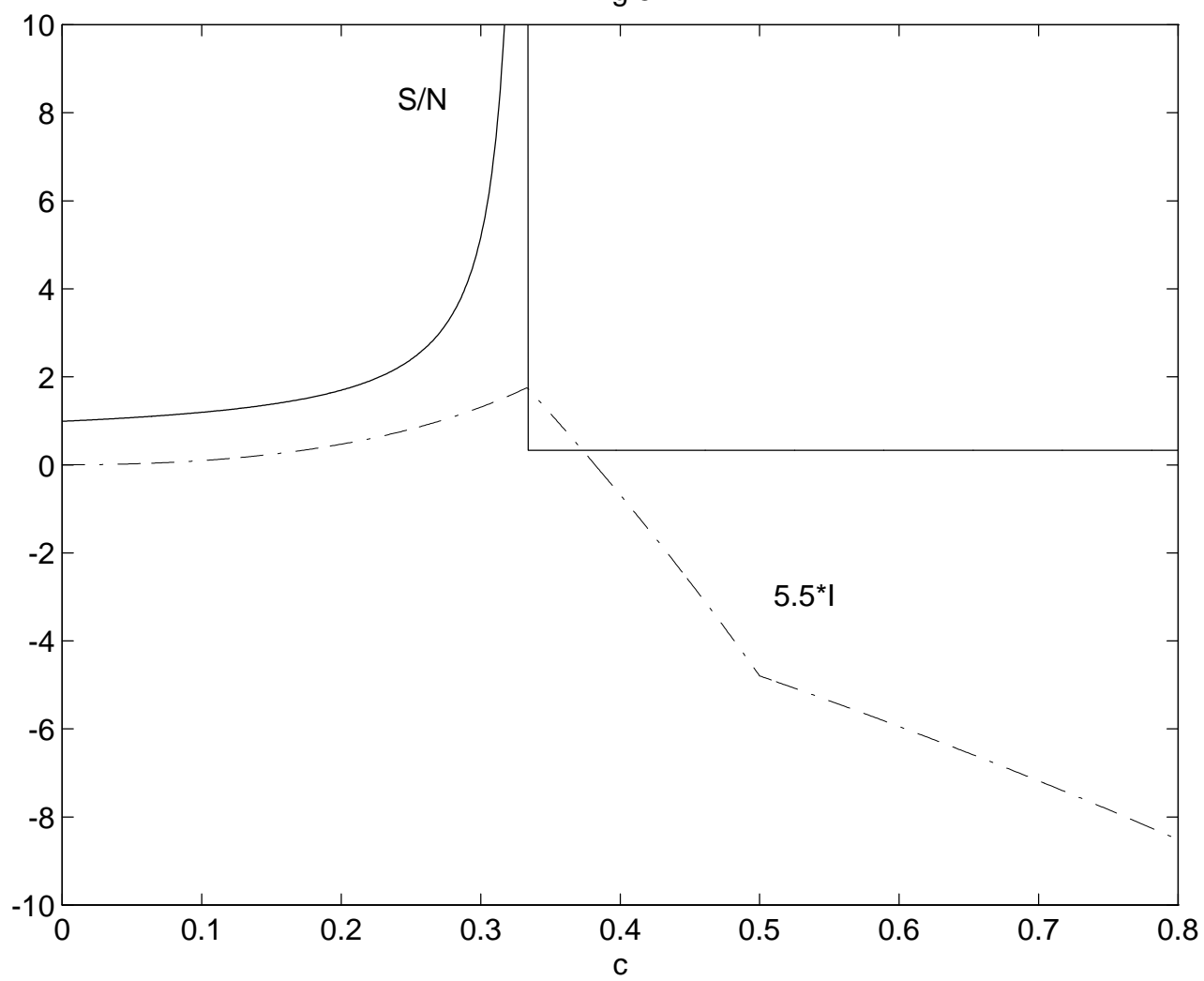
Fig.3

Fig.4

Fig.5

Fig.6

Fig.7

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Fig.9